ODE MODELS FOR THE PARACHUTE PROBLEM*

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Abstract. Classroom discussions relating to the modelling of physical phenomena has experienced a resurgence in recent years. One problem that is accessible to students and mathematically tractable is the motion of skydiver. The basic "parachute problem" is formulated and solved. A simple analysis of the problem raises questions about the applicability of this model. Real-life data is used to propose an extended model. The modified model is shown to be more physically realistic and is no more complicated than the original problem. This discussion gives equal emphasis to both the modelling and the analysis of the problem.

Key words. modelling, IVPs for ODEs, parachute problem, piecewise-defined data

AMS subject classifications. 34-01, 34A12

1. Introduction. The "parachute problem" is discussed in numerous differential equations textbooks (*e.g.*, [1, (p. 141, #19 and 20)], [3, (p. 95, #10, 11, 20, and 21)], [4, (p. 109, #20)], [7, (p. 112–114, Example 3 and #8)]) and journal articles (*e.g.*, [2], [6]). The appeal of the parachute problem is a combination of the facts that the basic model (Newtonian mechanics with resistance) is relatively simple for students to understand and that working with piecewise-defined functions, with which many students have some difficulty, is good preparation for the future discussion of Laplace transform methods. A common formulation of the parachute problem is:

> A skydiver drops from a helicopter hovering at a specified height, x_0, above the ground and falls toward the Earth under the influence of gravity. Assume the force due to air resistance is proportional to the velocity of the parachutist, with different constant of proportionality when the chute is closed (free-fall) and open (final descent). Given the condition that determines when the chute is deployed, how long does the jump last?

Typical questions to be addressed in the analysis of the problem include:

- what are the terminal velocities of the different stages of the jump? what is the velocity when the chute is opened? at impact?
- what is the latest time that the parachute can be opened while keeping the impact velocity below a specified threshold?
- compare the motions for jumps when the parachute is opened after a fixed amount of time, at a specified altitude, and when a given velocity is attained.
- find the corresponding model with quadratic air resistance, with coefficients selected so that the pre- and post-deployment terminal velocities are the same as for the linear model; how do the two motions compare?

The purpose of this note is to present an analysis of the traditional parachute problem that coordinates graphical solutions with the theory for initial value problems for a system of first-order ODEs. While this initial discussion is rather elementary, it does emphasize a number of important points: parameters identification, dimensional analysis, verification of solutions. The problem becomes more interesting at the end of Section 2, when the accuracy of the model is considered. Information from an Air Force Academy Training Guide [8] is used, in Section 3, to derive and analyze an

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Textbook	k_1/m	k_2/m	t_d	x_0
Abell/Braselton	1/6	5/3	60sec	n/a
Drucker	4/15	4/3	$x(14) = 0$	1200ft
Edwards/Penney	3/20	3/2	20sec	10,000ft
Kostelich/Armbruster	1/5	≈ 1.56	$v(t_d) = 0.95v_\infty$	2000m
Nagle/Saff	1/5	7/5	60sec	4000m

TABLE 2.1

Coefficients of air resistance, deployment criteria, and jump height for the parachute problem.

improved model. Additional extensions of the problem are included in the concluding remarks in Section 4.

2. The Basic Model. The motion of the skydiver is governed by Newton's Second Law of Motion. Balancing the forces of acceleration, gravity, and air resistance yields the second-order (linear) initial value problem

$$mx'' = -mg - kx', \qquad x(0) = x_0, \qquad x'(0) = 0$$

where x is the height of the skydiver above the Earth's surface, m is the skydiver's mass, g is the gravitational constant, k is the coefficient of air resistance, and prime ($'$) denotes differentiation with respect to time. An equivalent first-order system for the position and velocity,[1] v, is

$$\begin{aligned} v' &= -g - \tfrac{k}{m}v, & v(0) &= 0, \\ x' &= v, & x(0) &= x_0. \end{aligned} \tag{2.1}$$

This IVP has the advantage that the velocity equation $(2.1)_1$ can be solved as a first-order linear ODE; the position is then obtained by integration. Almost any ODE text contains the explicit solution for the case when m, g, and k are constants:

$$v(t) = \frac{mg}{k}\left(e^{-\frac{k}{m}t} - 1\right), \qquad x(t) = x_0 + \frac{m^2 g}{k^2}\left(-\frac{k}{m}t + \left(1 - e^{-\frac{k}{m}t}\right)\right). \tag{2.2}$$

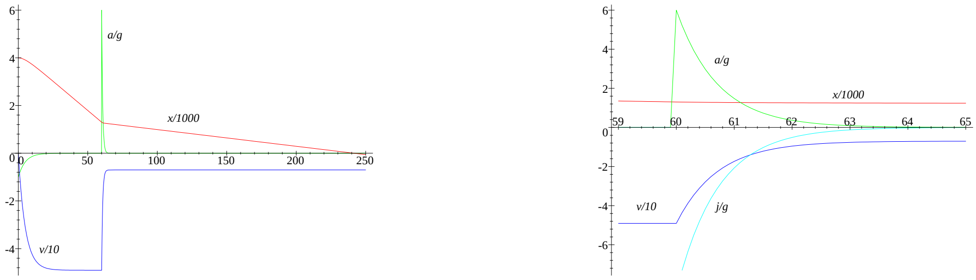
But, k is not constant in the parachute problem. The statement of the problem suggests the general form for the coefficient of air resistance is:

$$k(t) = \begin{cases} k_1, & t < t_d \\ k_2, & t \geq t_d \end{cases} \tag{2.3}$$

where t_d is the time when the parachute is deployed. The form of the differential equation suggests that k/m, with units 1/time, can be considered in place of the two parameters k and m. Note that this observation eliminates many of the potential problems that arise from the mixing of the CGS and MKS systems. Typical parameter values found in several ODE textbooks are reported in Table 2.1.

Note that if t_d is a function of velocity, *e.g.*, deployment occurs when the velocity reaches a specified threshold, or position, *e.g.*, deployment occurs at a given altitude, then the IVP is nonlinear. Typically, it will be necessary to find the solution with $k = k_1$, compute t_d, then solve the problem with $k = k_2$ and initial conditions selected to enforce continuity of position and velocity at the time of deployment: $v(t_d^+) = v(t_d^-)$

[1] Note that $v < 0$ for a body falling towards Earth.



(a) $0 \le t \le 250$ seconds (b) $59 \le t \le 65$ seconds

FIG. 2.1. *Motion for (a) $0 \le t \le 250$ seconds and (b) $59 \le t \le 65$ seconds for a piecewise constant coefficient of air resistance.*

and $x(t_d^+) = x(t_d^-)$ where $x(t_d^\pm)$ denote the right- and left-hand limits of the position at t_d, respectively. Even though the calculation is rather tedious, the explicit solution is not difficult to obtain;[2] for $t > t_d$ the velocity is

$$v(t) = \frac{mg}{k_1}\left(e^{-\frac{k_1}{m}t_d} - 1\right)e^{-\frac{k_2}{m}(t-t_d)} + \frac{mg}{k_2}\left(e^{-\frac{k_2}{m}(t-t_d)} - 1\right). \tag{2.4}$$

For the sake of this discussion, consider the parameters found in Example 3, p. 112, of Nagle and Saff [7]. Figure 2.1 shows graphical solutions for the position, velocity, and acceleration for 250 seconds after the jump begins as well as a closer look at the motion near the time the chute is deployed.

The graphical solution provides approximate answers to many of the simple questions about the jump. For example, the parachute is opened at an altitude of $\approx$ 1300m when the velocity is $\approx$ 49 m/s. Landing occurs a little more than three minutes later, with a velocity of $\approx$ 7 m/s. More precise answers can be found using the explicit solution (see [5]). Is an impact velocity of $\approx$ 7 m/s safely survivable without injury?

The degree to which the position appears to be piecewise linear is noteworthy, particularly when compared to the complicated form of the exact solution, $(2.2)_2$ and the integral of (2.4). This is easily explained by the rapid convergence of the velocity to the terminal velocity during each stage of the jump. The graph of the velocity immediately following deployment of the chute is so steep that the curious reader might question whether the curve is continuous. (This is a good question for the students.) But, what does this say about the acceleration?

Notice that computing the acceleration by differentiation, $a = v'$, is complicated by the piecewise definition of the velocity. (What is $v'(t_d)$?) A simpler method of obtaining the acceleration is to refer directly to the ODE that governs the velocity $(2.1)_1$. That is, $a = -g - \frac{k}{m}v$. This approach to the acceleration clearly indicates that there is a jump in the acceleration at the instant the parachute is deployed. The *snatch force* is the acceleration at the first instant when the assembly reaches full extension; the *opening shock*, or *jerk*, is the shock produced while the parachute deploys [8].

[2]This is a perfect situation to use a computer algebra system. See [5] for a discussion of the use of Maple for the parachute problem.

Mathematically, the jerk is the time derivative of the acceleration,

$$j(t) := \frac{da}{dt} = -\frac{k'(t)}{m}v(t) - \frac{k(t)}{m}a(t). \tag{2.5}$$

When k is piecewise constant, as in (2.3), the snatch force has a jump discontinuity at the instant of deployment: $[j(t_d)] := j(t_d^+) - j(t_d^-) = -\frac{[k(t_d)a(t_d)]}{m}$. Using the Nagle/Saff parameters, $a(60^-) \approx 0$ G, $a(60^+) \approx 6$ G, and $[j(60)] \approx -8.4$ G/s. These jumps seem to be fairly significant; are they realistic?

3. Real-World Considerations. The preceding discussion illustrates techniques used to answer a wide variety of mathematical and physical questions about ODE models. The analysis raised several additional questions about the applicability of the results to a real-life parachute jump. To address these issues it is necessary to obtain real-life data about skydiving; two accessible references for this material include [8] and [9].

Training jumps for the Parachute Team at the United States Air Force Academy begin 4,000′ (1,219 m) above ground level (AGL). The free-fall portion of the jump lasts about 10 seconds; free-falls longer than 13 seconds are grounds for removal from the team. The terminal velocity for free-fall is 120 miles/hr (176 ft/sec or 53.6 m/sec). The parachute requires approximately 3.2 seconds to fully deploy from the time the ripcord is pulled — at an altitude of at least 2,500′ (762 m) AGL. The snatch force felt when the lines and canopy are fully elongated is a heavy, but smooth, tug that is not particularly uncomfortable. While this force depends on the weight of the skydiver, it should not exceed 500 lbs (≈3 G for a 165 lb (75 kg) person). The harness and parachute are designed to withstand a force of 5000 lb (30 G). The landing velocity should be no worse than a free-fall from a 5′ (1.52 m) wall — between 15 and 17 ft/sec (4.6 and 5.2 m/sec). A reserve chute is required on all intentional jumps. If a malfunction occurs with the main chute at 3000′ (912 m) AGL, almost 6 seconds will be required to recognize and react to the problem and to deploy the reserve chute. The reserve chute must be opened no lower than 1000′ (304 m) AGL; deployment requires only 1.5 seconds and the landing velocity should not exceed 17.5 ft/sec (5.3 m/sec).

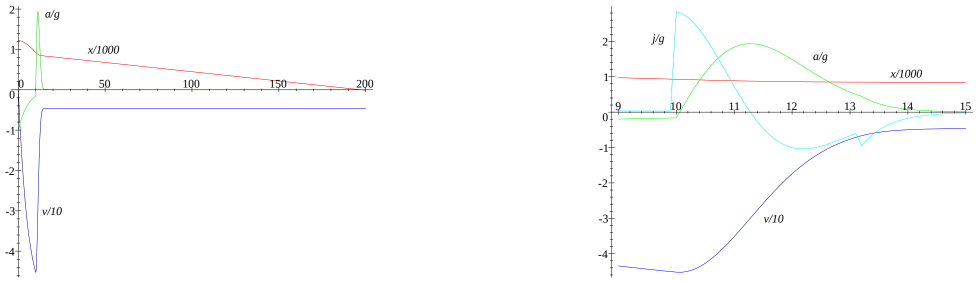
A free-fall terminal velocity of 120 miles/hr implies $\frac{k_1}{m} = \frac{2}{11}$ and a landing velocity of 16 ft/sec corresponds to $\frac{k_2}{m} = \frac{32}{16}$. Note that the values of $\frac{k_1}{m}$ found in Table 2.1 are relatively realistic, the values of $\frac{k_2}{m}$ are uniformly low (so that the landing velocities are too high). The jump height and deployment time can be more varied; constraints on these parameters arise from *e.g.*, the minimum height needed for the parachute to deploy and the applicability of the model at higher altitudes.[3]

The statement that the snatch force should be smooth, and not exceed 3 G, suggests that (2.3) is not appropriate. A smooth snatch force is assured only when $k \in C^1(0, \infty)$. The fact that the parachute does not open instantaneously provides the necessary flexibility to connect the two constant states in a continuous manner. The resulting form for k is

$$k(t) = \begin{cases} k_1, & t < t_d \\ k_d, & t_d \le t < t_d + \tau_d \\ k_2, & t \ge t_d + \tau_d \end{cases} \tag{3.1}$$

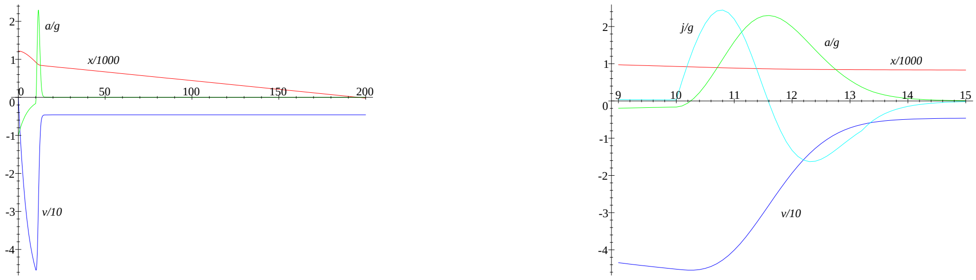
where the time required for the parachute to open is denoted by τ_d. Note that an explicit solution to (2.1) with k as in (3.1) can be obtained by solving three IVPs.

[3] It should be noted that some of the textbook problems become more realistic simply by changing the units.



(a) $0 \le t \le 200$ seconds (b) $9 \le t \le 15$ seconds

FIG. 3.1. *Motion for (a)* $0 \le t \le 200$ *seconds and (b)* $9 \le t \le 15$ *seconds for a* C^0 *coefficient of air resistance.*



(a) $0 \le t \le 200$ seconds (a) $9 \le t \le 15$ seconds

FIG. 3.2. *Motion for (a)* $0 \le t \le 200$ *seconds and (b)* $9 \le t \le 15$ *seconds for a smooth* (C^1) *coefficient of air resistance.*

Assuming t_d and τ_d are known, or can be determined, each problem is a first-order linear ODE. In this sense the modified model is no more complicated that the original.

To complete the model it is necessary to choose a specific k_d. Continuity of the acceleration will be assured when k_d interpolates the two constant states. For example, when k_d is a linear function, the motion (position, velocity, acceleration, and jerk) appear as in Figure 3.1. Note that this model predicts the maximum jerk is under the 3 G threshold, but occurs at the instant the ripcord is pulled. To obtain a smooth jerk k_d must be selected so that the derivatives at $t = t_d$ and $t = t_d + \tau_d$ both vanish. A cubic function can easily be fit to these conditions; the resulting motion is displayed in Figure 3.2. Note that this motion is consistent with all of the characteristics found in the Air Force Academy Training Manual [8]: snatch force is smooth and well below the 3 G threshold; deployment begins at an altitude of 928 m (3045′) AGL; an additional 3 seconds of free-fall brings the altitude to ≈2500′ — thus the strict penalty for free-falls longer than 13 seconds. The complete jump lasts a little more than three minutes (196 seconds) with a landing velocity of 4.6 m/s (15.1 ft/sec).

4. Conclusion. This note identifies a number of concerns about the parachute problem as it appears in several ODE texts. Real-life data is presented and used to create an improved model that can be analyzed using similar methods. The predictions obtained from the new model are consistent with the physics of skydiving. The modified model is still relatively simple. For example, while the true motion is three-dimensional, the current models consider only the vertical component of the motion. Further extensions of the model make good project assignments.

For jumps that begin at altitudes higher than $25{,}000'$ above sea level it begins to be reasonable to consider including the altitude dependence of the air density, air pressure, and gravitational constant in the model. A simple investigation of the sensitivity of the solution to the different parameters is useful when deciding which (if any) of the parameters should be allowed to be altitude dependent (see [5]).

Another extension of the problem is to consider models with nonlinear (quadratic) air resistance during one or more of the three stages of the jump. While the numerical and graphical analysis can proceed virtually unchanged, explicit solutions can be more difficult to obtain (manually) since the individual IVPs become (nonlinear) Bernoulli equations.

The list of physically interesting situations that can be analyzed is almost endless. One obvious example is to formulate a model consistent with the information about the reserve chute and to check that the stated constraints can be satisfied. (What parameters are appropriate for descent under the reserve chute? How smooth is the jerk? Can the cubic model for k_d be used?) Other sets of problems can be constructed around different criteria for chute deployment: *e.g.*, at a specified velocity or altitude. Two control problems that can be particularly instructive are: Given a desired duration of the jump, when should the ripcord be pulled so that the landing occurs at the desired time? and What is the latest time the ripcord can be pulled so that the landing velocity is below a given threshold?

The computations and graphs in this paper were prepared using Maple. Full details, including solutions to some of the problems posed above, can be found in [5].

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